A Quick View of Fourier Optics

The intensity of light falling on a surface can be represented as the square of a supposed real wave-function:

$$I(\vec{r},t) = 2\langle u^2(\vec{r},t)\rangle$$

Where the real wave-function is the real part of a complex wave-function:

$$u(\vec{r},t) = \frac{1}{2}\left[U(\vec{r},t) + U^*(\vec{r},t)\right]$$

and the complex wave-function consists of a complex amplitude times the harmonic function in time representing the frequency of the light (monochromatic for now):

$$U(\vec{r},t) = U(\vec{r})e^{i\omega t}$$

The time independent factor is the complex amplitude and will consist of a harmonic function along the axis of travel times an arbitrary amplitude function (arbitrary as long as it satisfies the Helmholtz equation):

$$U(\vec{r}) = a(\vec{r})e^{i\vec{k}\cdot\vec{r}}$$

or

$$U(\vec{r}) = a(\vec{r})e^{i\phi(\vec{r})}$$

Lets start with a plane wave with complex amplitude:

$$U(x,y,z) = A\exp\left[-i\left(k_x x + k_y y + k_z z\right)\right]$$

with the wave-vector $\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ and $\left|\vec{k}\right| = k = \sqrt{k_x^2 + k_y^2 + k_z^2} = 2\pi/\lambda$.

This wave-vector makes angles with the y,z and x,z planes:

$$\theta_x = \sin^{-1}\left(k_x/k\right)$$
$$\theta_y = \sin^{-1}\left(k_y/k\right)$$

The paraxial approximation holds if the z component of k is much larger than either of the other two components.

If we cut through this plane wave with constant z plane and call that location $z = 0$, then the cross section impinging on this plane is the complex amplitude profile of the plane wave on this (x,y) plane. (Note that the square of this complex amplitude gives the intensity profile on this (x,y) plane and in the case of the plane wave it is uniform illumination).

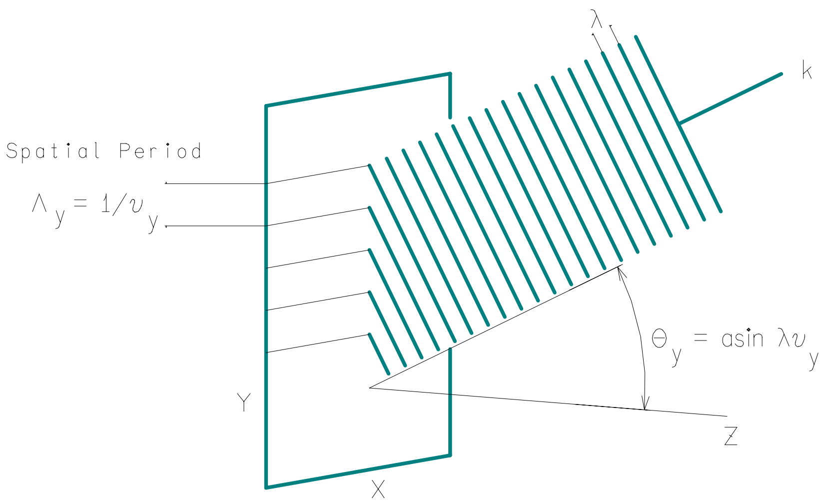
The complex amplitude on the $z = 0$ plane:

$$U(x,y,0) = f(x,y) = A\exp\left[-i2\pi\left(\nu_x x + \nu_y y\right)\right]$$
$$\nu_x = k_x/2\pi$$
$$\nu_y = k_y/2\pi$$

Here, ν_x and ν_y are the **spatial frequencies of the complex amplitude** along the x and y directions on the $z = 0$ plane.

Note now that if the paraxial approximation applies, then:

$$\theta_x \approx \lambda \nu_x$$
$$\theta_y \approx \lambda \nu_y$$



As a side note a hologram or a diffraction grating functions by imposing a complex amplitude transmittance at a certain plane so that the transmitted wave has this function $f(x, y)$ forced on it at that plane. A plane wave in the z direction becomes a plane wave in the z direction plus plane waves at the appropriate angles such that the forced spatial frequencies apply. Since these spatial frequencies, which are now forced by transmission properties, would have depended on both angle and frequency of an incoming wave, the angle of the transmitted wave depends on the temporal frequency of the wave and thus a diffraction grating acts to separate different wavelength components of a beam into beams in different directions.

Now suppose the complex transmittance imposed on a plane wave, or equivalently, the complex amplitude cross section of a light beam, $f(x, y)$, is a superposition of many spatial frequencies (for example a combination of spatial frequencies making up a photograph).

$$f(x, y) = \iint F(\nu_x, \nu_y) \exp\left[-i2\pi\left(\nu_x x + \nu_y y\right)\right] d\nu_x d\nu_y$$

Where $F(\nu_x, \nu_y)$ is the amplitude of the (ν_x, ν_y) spatial frequency combination. Then the transmitted light (or the light moving beyond this plane in the beam) can be decomposed into a superposition of plane waves such that each combination of spatial frequencies, (ν_x, ν_y), is represented by a plane wave moving in a unique direction. In this view, $F(\nu_x, \nu_y)$ is the complex envelope of the plane wave component of the beam moving in that unique direction. The intensity profile of the light on a given plane is the square of this resultant complex amplitude function.

$$I(x,y,z) = |U(x,y,z)|^2$$

$$U(x,y,z) = \iint F(\nu_x, \nu_y)\exp[-i(2\pi\nu_x x + 2\pi\nu_y y)]\exp(-ik_z z)d\nu_x d\nu_y$$

$$= \iint F(\nu_x, \nu_y)\exp(-ik_z z - ik_x x - ik_y y)d\nu_x d\nu_y$$

Seen this way, free space propagation of a beam acts as a "spatial prism" separating the spatial frequency components of a beams complex profile just as a prism separates the temporal frequency components.

The function of a lens places the energy of a plane wave at a particular location on the focal plane which location is determined by the direction of the plane wave as it impinges on the lens. Since we have a unique direction for every spatial frequency component of a beam then we would have a unique location on the focal plane of a lens for each spatial frequency component of the beam falling on the lens. The lens maps each direction, (θ_x, θ_y) into a single point $(x,y) = (\theta_x f, \theta_y f) \approx (\lambda f \nu_x, \lambda f \nu_y)$.

If the complex amplitude of the beam falling on the lens is: $f(x,y)$ then the amplitude of a given spatial frequency pair (ν_x, ν_y) and thus a given direction (θ_x, θ_y), is the Fourier transform of this complex amplitude:

$$f(x,y) = \iint F(\nu_x, \nu_y)\exp[-i2\pi(\nu_x x + \nu_y y)]d\nu_x d\nu_y$$

$$\Updownarrow$$

$$F(\nu_x, \nu_y) = \iint f(x,y)\exp[i2\pi(\nu_x x + \nu_y y)]dxdy$$

Then the complex amplitude of the beam on the focal plane is given by:

$$g(x,y) \propto F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$$

This means that the complex amplitude on the focal plane is proportional to the Fourier transform of the complex amplitude of the beam falling on the lens.

If we begin with an input plane a distance d from the lens, then the proportionality factor can be calculated from the transfer function of the free space traveled and the transfer function of the lens. The final result is that the complex amplitude on the back focal plane at the position (x,y) is proportional to the Fourier transform of the complex amplitude on the input plane a distance d in front of the lens evaluated at the frequencies $\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$:

$$g(x,y)=\frac{i}{\lambda f}e^{-i2kf}e^{\left[i\pi\frac{\left(x^2+y^2\right)(d-f)}{\lambda f^2}\right]}F\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$$

$$=\frac{i}{\lambda f}e^{-i2kf}e^{\left[i\pi\frac{\left(x^2+y^2\right)(d-f)}{\lambda f^2}\right]}\iint f(x',y')\exp\left[i2\pi\left(\frac{x}{\lambda f}x'+\frac{y}{\lambda f}y'\right)\right]dx'dy'$$

Note that the resultant intensity is independent of the distance d:

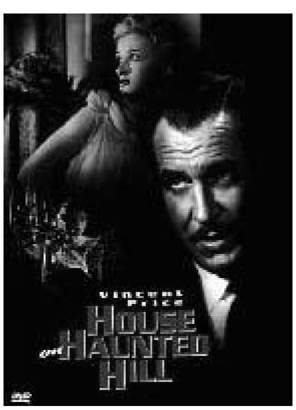
$$I(x,y)=\frac{1}{(\lambda f)^2}\left|F\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)\right|^2$$

If we set the distance d equal to the front focal length we simplify the equation a little further:

$$g(x,y)=\frac{i}{\lambda f}e^{-i2kf}F\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)$$

Now suppose we put a screen at the back focal length, and screen out some specific spatial frequencies, then put another lens one focal length beyond this screen. On the back focal plane of this new lens we will re-image the first plane one focal length in front of the first lens but we will have filtered out specific spatial frequency combinations. For example, a screen blocking the central portion near the axis of the system will screen out low spatial frequencies and the result is an image outlining the high contrast locations of the original. This is a High Pass filter. A filter that consists of a small hole around the center portion and everything else blocked acts as a Low Pass filter blocking all of the high spatial frequencies and resulting in a softened image compared to the original that will have slower fluctuation over the image plane. This situation corresponds to the pinhole filtering of a laser beam in a beam expansion telescope.

Original Image



Low Pass Filtered

High Pass Filtered